Calculation of electromagnetic field components for a fundamental Gaussian beam

Guozhong Wang[1] and J. F. Webb[2]
[1]Department of Physics, Wenzhou Normal College, Wenzhou 325027, People's Republic of China
[2]Sirindhorn International Institute of Technology, Thammasat University, P. O. Box 22, Thammasat Rangsit Post Office, Pathumthani 12121, Thailand
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Using a perturbative method, we carefully calculate seventh- and ninth-order expressions for electromagnetic field components for a fundamental Gaussian beam (i.e., a focused TEM_{00} mode laser beam) propagating in the medium analogous to vacuum. The interaction of a single electron and a focused laser pulse in vacuum has a promising application in the design of a powerful electron accelerator which could compete with the traditional ones. For this, ninth-order accuracy in the expansion describing the focused laser beam is required. When the vector potential corresponds to polarization along the direction of propagation, the number of electromagnetic field components can be reduced from 5 or 6 to 3. Furthermore, we find two rules obeyed by all orders of the vector potential, which greatly simplify the calculation procedure of the vector potential order by order, and make it possible to investigate the behavior of high orders.

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I. INTRODUCTION

State-of-the-art petawatt lasers can deliver very high energy pulses that can be focused to an intensity of as high as 10^{21} W/cm^2 [1], corresponding to a 10 TV/m optical field [2]. An important application of such strong laser pulse is to make a very compact accelerator with GeV or TeV energies. Several experiments have successfully shown the feasibility of producing energy gains of elementary particles from ultraintense laser beams [3–6]. The dynamics of charged particles interacting with the electromagnetic field of a laser beam is a central theoretical problem in this area. A detailed knowledge of the electromagnetic field of the laser pulse near the beam focus is required. In the plane perpendicular to the propagating direction, the intensity of Gaussian beam decays exponentially with the distance from the propagating axis, which can be used to describe the focused laser beam. The Gaussian beam has the advantage over other beam geometries in that it usually constitutes the fundamental laser mode, and is therefore generally available. The fundamental Gaussian beam can precisely describe the axially symmetrical focused laser beam, and has been used often in the theoretical treatment of laser induced electron acceleration [7–9]. To our knowledge, the highest order description referred to the small parameter s $(1/kr_0)$ of the electromagnetic field for a fundamental Gaussian beam is 5 [10–12]. The smaller the beam waist, the higher the order of correction that is required to guarantee the reliability of the calculation. Salamin *et al.*'s simulation results show that when the beam waist is smaller than 7 μm, the energy gains of electrons in an ultraintense Gaussian beam described by third- and fifth-order terms in s differ greatly [8], which means that a fifth-order description of the fundamental Gaussian beam will not be adequate for a tightly focused beam. Furthermore, most numerical simulations have not included nonlinear effects, such as the radiation of the charged particle and the ponderomotive force [13–15]. So, it is evident that a more precise calculation of the dynamics of a tightly focused laser beam and charged particle interaction that would be able to predict the particle trajectory and energy gain would require a description of a Gaussian beam beyond fifth order.

II. CALCULATION

In this paper, the method introduced by Davis [11] is used to derive seventh- and ninth-order corrected expressions for the electromagnetic field components of a fundamental Gaussian beam. We consider the propagation of a monochromatic beam within an isotropic, homogeneous, nonmagnetic ($\mu=1$), and nonconducting ($\sigma=0$) dielectric medium which especially denotes vacuum for ultraintense laser beam, and a harmonic time dependence of $\exp(i\omega t)$ ($i=\sqrt{-1}$ throughout this paper) is assumed and dropped from all subsequent time-dependent terms. Under these conditions, Maxwell's equations can be written, in Gaussian unit, as

$$\nabla \cdot \mathbf{E} = 0, \tag{1}$$

$$\nabla \times (\mathbf{H}/\sqrt{\varepsilon}) - ik\mathbf{E} = \mathbf{0}, \tag{2}$$

$$\nabla \times \mathbf{E} + ik(\mathbf{H}/\sqrt{\varepsilon}) = \mathbf{0}, \tag{3}$$

$$\nabla \cdot (\mathbf{H}/\sqrt{\varepsilon}) = 0, \tag{4}$$

where $k=\sqrt{\varepsilon}\omega/c=2\pi/\lambda$ is the wave number for the medium, and ε the dielectric constant of the medium. In Lorentz gauge, a vector potential $\mathbf{A}$ can be defined such that if $\mathbf{A}$ is a solution of the Helmholtz equation

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = \mathbf{0}, \tag{5}$$

then

$$\mathbf{E} = -(i/k)\,\nabla(\nabla \cdot \mathbf{A}) - ik\mathbf{A} \tag{6}$$

and

$$\mathbf{H}/(\sqrt{\varepsilon}) = \nabla \times \mathbf{A} \tag{7}$$

satisfy Maxwell's equations [Eqs. (1)–(4)]. Considering a beam propagating along the z axis and polarized along the x

axis, we can set $\mathbf{A}=A\hat{\mathbf{x}}$, where $A=\psi(x,y,z)\exp(-ikz)$. There exist two parameters with the dimensions of length (the beam waist radius r_0 and the diffraction length $l=kr_0^2$) which can be used to normalize the spatial coordinates: $\xi=x/r_0$, $\eta=y/r_0$, and $\zeta=z/l$. The vector potential then becomes $A=\psi(\xi,\eta,\zeta)\exp(-i\zeta/s^2)$, and the Helmholtz equation, in terms of ψ, is

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}-2i\frac{\partial}{\partial\zeta}\right)\psi=-s^2\frac{\partial^2\psi}{\partial\zeta^2}, \tag{8}$$

where $s=1/kr_0$ is assumed to be a small parameter, so that ψ can be expanded as a sum of even powers of s

$$\psi=\sum_0^n s^{2n}\psi_n, \tag{9}$$

in which ψ_n satisfies

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}-2i\frac{\partial}{\partial\zeta}\right)\psi_0=0, \tag{10}$$

for $n=0$ and

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}-2i\frac{\partial}{\partial\zeta}\right)\psi_{2n+2}=-\frac{\partial^2\psi_{2n}}{\partial\zeta^2}, \tag{11}$$

for $n>0$. Here, ψ_0 is the familiar paraxial solution, and ψ_2 and ψ_4 have been found by Davis [11], and Barton and Alexander [12]. They are

$$\psi_0=iq\exp(-iq\rho^2),$$

$$\psi_2=(2iq+iq^3\rho^4)\psi_0,$$

$$\psi_4=(-6q^2-3q^4\rho^4-2iq^5\rho^6-0.5q^6\rho^8)\psi_0,$$

where $\rho^2=\xi^2+\eta^2$, and $q=1/(i+2\zeta)$.

As a first step in calculating ψ_6 and ψ_8, their forms are found using the systematic method adopted by Davis [11]. In this method the paraxial approximation for a diverging spherical wave propagating from the origin can serve as a guide. Formally, we have

$$e^{-ik\sqrt{z^2+r^2}}=e^{-ikz-iq\rho^2}\sum_{n=0}^{\infty}s^{2n}f_{2n}, \tag{12}$$

where $f_0=1$, and

$$f_2=iq^3\rho^4,$$

$$f_4=-(2iq^5\rho^6+q^6\rho^8/2),$$

$$f_6=5iq^7\rho^8+2q^8\rho^{10}-iq^9\rho^{12}/6,$$

$$f_8=-(14iq^9\rho^{10}+7q^{10}\rho^{12}-iq^{11}\rho^{14}-q^{12}\rho^{16}/24).$$

For this the relation $q\to1/2\zeta=l/2z$ for large ζ has been used to eliminate the variable z. From the expression for f_6 above, ψ_6 has the form

$$\psi_6=[(5iq^7\rho^8+2q^8\rho^{10}-iq^9\rho^{12}/6)+a(q)\rho^8+b(q)\rho^6+c(q)\rho^4+d(q)]\psi_0. \tag{13}$$

All the coefficients $a(q)\cdots d(q)$ can be determined from Eq. (11), and the result is

$$a(q)=-2iq^7,\quad b(q)=8q^6,\quad c(q)=-12iq^5,$$

$$d(q)=-24iq^3. \tag{14}$$

Repeating the same process gives

$$\psi_8=(-q^{12}\rho^{16}/24+iq^{11}\rho^{14}-37q^{10}\rho^{12}/6-4iq^9\rho^{10}+q^8\rho^8+296iq^7\rho^6-1092q^6\rho^4+120q^4)\psi_0. \tag{15}$$

Using $A\doteq\exp(-ikz-i\zeta/s^2)(\psi_0+\cdots+s^8\psi_8)$, the electromagnetic field components to ninth order can be calculated, because $\mathbf{A}=A\hat{\mathbf{x}}$ and $H_x=0$ for all orders of s. To construct a symmetric electromagnetic field, let the coordinate system rotate by $\pi/2$ about the z axis; the corresponding rotation matrix is

$$R=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

The coordinates change from $\mathbf{r}=(x,y)^T$ to $\mathbf{r}=(y,-x)^T$, and the electromagnetic fields satisfy the rule for vector function transformation, namely

$$(E_1',E_2')^T=R(E_1'(R^{-1}r),E_2'(R^{-1}r))^T, \tag{16}$$

corresponding to the vector potential changing from $A\hat{\mathbf{x}}$ to $A\hat{\mathbf{y}}$. Combining the dual symmetry, $\mathbf{H}\to\mathbf{E}/\sqrt{\varepsilon}$ and $\mathbf{E}\to-\mathbf{H}$, of Maxwell's equations [Eqs. (1)–(4)], we get new electromagnetic field components

$$E_1''=H_2(-y,x,z),\quad E_2''=-H_1(-y,x,z),\quad E_3''=H_3(-y,x,z), \tag{17}$$

and the magnetic field components transform to

$$H_1''=-\sqrt{\varepsilon}E_2(-y,x,z),\quad H_2''=\sqrt{\varepsilon}E_1(-y,x,z),$$

$$H_3''=-\sqrt{\varepsilon}E_3(-y,x,z). \tag{18}$$

Two sets of solutions added and divided by 2 result in symmetric electric field components, apart from a common factor $E_0\psi_0\exp(-i\zeta/s^2)$ (here, E_0 is the electric field amplitude at the focal point of the beam). These solutions are

$$E_x=1+(-2q^2\xi^2-q^2\rho^2+iq^3\rho^4)s^2+[(8q^4\rho^2-2iq^5\rho^4)\xi^2+2q^4\rho^4-3iq^5\rho^6-q^6\rho^8/2]s^4+[(-30q^6\rho^4+12iq^7\rho^6+q^8\rho^8)\xi^2-5q^6\rho^6+9iq^7\rho^8+5q^8\rho^{10}/2-iq^9\rho^{12}/6]s^6+[(112q^8\rho^6-56iq^9\rho^8-8q^{10}\rho^{10}+iq^{11}\rho^{12}/3)\xi^2-1152q^6\rho^4+256iq^7\rho^6+30q^8\rho^8-28iq^9\rho^{10}-10q^{10}\rho^{12}+7iq^{11}\rho^{14}/6+q^{12}\rho^{16}/24)s^8], \tag{19}$$

$$E_y=[-2q^2s^2+(8q^4\rho^2-2iq^5\rho^4)s^4+(-30q^6\rho^4+12iq^7\rho^6+q^8\rho^8)s^6+(112q^8\rho^6-56iq^9\rho^8-8q^{10}\rho^{10}+iq^{11}\rho^{12}/3)s^8]\xi\eta, \tag{20}$$

TABLE I. Average and maximal value of ratios calculated [see Eqs. (23) and (24)] for fifth- to ninth-order descriptions of a Gaussian beam.

$s=$	0.02	0.05	0.10	0.20	0.30	0.40
s^5 avg	1.56×10^{-9}	3.80×10^{-7}	2.40×10^{-5}	1.47×10^{-3}	0.015	0.065
max	2.85×10^{-8}	6.93×10^{-6}	4.32×10^{-4}	3.37×10^{-2}	0.547	1.43
s^7 avg	4.23×10^{-12}	6.44×10^{-9}	1.63×10^{-6}	4.10×10^{-4}	0.00896	0.0890
max	1.10×10^{-10}	1.67×10^{-7}	4.17×10^{-5}	1.39×10^{-2}	0.417	7.21
s^9 avg	4.95×10^{-14}	4.70×10^{-10}	4.75×10^{-7}	4.69×10^{-4}	0.0289	0.253
max	3.53×10^{-13}	3.41×10^{-9}	3.70×10^{-6}	5.49×10^{-3}	1.47	1.86

$$E_z=[-2qs+(6q^3\rho^2-2iq^4\rho^4)s^3+(-20q^5\rho^4+10iq^6\rho^6+q^7\rho^8)s^5+(70q^7\rho^6-42iq^8\rho^8-7q^9\rho^{10}+iq^{10}\rho^{12}/3)s^7+(4608iq^6\rho^2+3840q^7\rho^4-640iq^8\rho^6-284q^9\rho^8+168iq^{10}\rho^{10}+36q^{11}\rho^{12}-3iq^{12}\rho^{14}-q^{13}\rho^{16}/12)s^9]\xi. \tag{21}$$

The corresponding symmetric magnetic field components are $H_x/\sqrt{\varepsilon}=E_y(\xi\rightarrow\eta)$, $H_y/\sqrt{\varepsilon}=E_x(\xi\rightarrow\eta)$, and $H_z/\sqrt{\varepsilon}=E_z(\xi\rightarrow\eta)$.

From these electromagnetic field components, we can calculate the energy flux F averaged over a period of the laser beam by integrating Poynting vector at $z=0$ plane

$$F=\frac{1}{2}\int \mathrm{Re}(E^*\times H)d\vec{\sigma}=\frac{\sqrt{\varepsilon}|E_0|^2\pi r_0^2}{8}(2+2s^2+3s^4+6s^6+1647s^8), \tag{22}$$

which can make the electromagnetic amplitude E_0 relate to the beam power or intensity. The sudden increase of the coefficient of s^8 in Eq. (22) suggests that the perturbative solution is asymptotic.

III. ACCURACY

The ninth-order corrected Gaussian beam description was verified by directly substituting the electromagnetic field components into Maxwell's equations [Eqs. (1)–(4)]. Ratios can be calculated by taking the deviation of the magnitude of the left-hand side of Maxwell's equations from zero to $k|\mathbf{E}|$ for Eqs. (1) and (2) and relative to $k|\mathbf{H}|/\sqrt{\varepsilon}$ for Eqs. (3) and (4). Because of the symmetry of Maxwell's equations and the calculated electromagnetic field of the laser beam, ratios were only calculated for Eqs. (1) and (2). Since, for a laser beam in vacuum, $\varepsilon=1$, we calculate the quantities

$$\frac{|\nabla\cdot\mathbf{E}|}{k|\mathbf{E}|}, \tag{23}$$

and

$$\frac{|\nabla\times\mathbf{H}-ik\mathbf{E}|}{k|\mathbf{E}|}. \tag{24}$$

Table I provides a comparison of these ratios at $s=0.02,0.05,0.10,0.20,0.30,0.40$ for fifth- to ninth-order Gaussian beam descriptions. Both the average and the maximum ratios were calculated for 216 spatial positions surrounding the focal point consisting of all combinations of $\xi,\eta,\zeta=0.0,0.1,0.2,0.5,1.0$ (for comparison, we chose the same values as those in Ref. [12]). The ninth-order description gives a significant improvement in accuracy. We note that our calculated values for fifth order do not completely match those of Ref. [12]. The possible reason is the difference between the formulas used for the calculation here and those in Ref. [12]. The data listed in Table I are calculated from Eq. (24), because the corresponding data calculated from Eq. (23) are all smaller than those calculated from Eq. (24).

IV. ELECTRON ACCELERATION BY LASER PULSE

Accelerating electrons to extremely high energy is one of the most important applications of ultraintense laser pulses, and is an active area of research. According to whether or not a plasma is involved, the laser-driven electron acceleration schemes can be classified into two kinds: acceleration in plasma and acceleration in vacuum. When a laser pulse is injected into a preformed plasma, its transverse electric field sets the electrons in plasma into transverse oscillation, making them gain energy, which corresponds to a momentum decrease of the laser pulse. Due to the momentum conservation of the laser pulse and electron (or in other words the electron feels the Lorentz force $\mathbf{v}\times\mathbf{H}$, which is also referred to as ponderomotive force), the electron would pick up a longitudinal momentum and be displaced, leading to the appearance of a space charge in the plasma which pulls electron back, and a plasma oscillation called the wake plasmon is set up. The phase velocity of the wake plasmon is the same as the group velocity of laser pulse propagating in the plasma, $v_p=c(1-\omega_p^2/\omega^2)^{1/2}$, where ω_p is the plasma frequency and ω the laser pulse frequency. The point is that v_p can be very close to the speed of light c, which largely determines the energy gain of the electron. The wake plasmon is a longitudinal wave which can trap background electrons and accelerate them to a high energy due to the longitudinal electric field whose typical value is $E_L\approx n_e^{1/2}$ V/m $\approx 10^{11}$ V/m for a laser pulse wavelength $\sim 1\ \mu$m and a plasma density $\sim 10^{18}$ cm^{-3}; here, n_e is the electron density in the plasma. The plasma acceleration is based on the longitudinal electric field of the wake plasmon

[16]. The plasma serves a dual purpose: it act as a medium which (a) transforms a fraction of the laser field into an acceleration field and (b) modifies the refractive index to optically guide the laser pulse. Even for a small-amplitude plasma wave, the refractive index is no longer constant. This applies, for example, to a self-modulated laser wake-field regime, where the laser pulse is much longer than the plasma wavelength λ_p, and the laser pulse envelope becomes periodically modulated at λ_p. In a forced laser wake-field regime, however, where the length of the laser pulse is about $\lambda_p/2$, the laser pulse is strongly compressed by group velocity dispersion: the front of the laser pulse pushes electrons forward and the rear propagates in the density depression of the plasma wave. Consequently, the back of the pulse propagates faster than the front, compressing it to create an optical shock. Because the longitudinal accelerating field of the plasmon is transformed from a laser pulse, and laser-plasma interaction is a many-body problem, a precise description of the initial laser pulse is not necessary. In fact, the laser-plasma interaction is relativistic and nonlinear, and should be described by a set of coupled differential equations [17,18]. The laser-plasma acceleration technique has made great progress. However, there are inherent defects. For instance, the relativistic mass increase of the oscillating electrons changes the plasma frequency, causing the resonance to be lost; this prevents the amplitude of the accelerating field from reaching its maximum determined by wave breaking. Also, the energetic electrons are divergent and continuous in the energy spectrum, with only a very small fraction of the electrons having a maximum energy. Defects such as these mean that accelerator based on laser-plasma interaction are unable to compete with the present high-energy linacs [19].

Laser-driven electron acceleration in vacuum can circumvent all the shortcomings concerned with laser-plasma interaction. The Lawson-Woodward (LW) theorem indicates that no net energy can be transferred by a beam to a particle if the interaction time is infinite [20]. However, by breaking the conditions under which the LW theorem holds, many schemes enabling energy to be extracted from the pulse have been put forward and verified [8,21,22]. The mechanism for laser-driven electron acceleration in vacuum is based on the interaction of a tightly focused laser pulse and electron. For simplicity, let us consider the interaction of a laser pulse and an incident electron in vacuum. The dynamics of the laser pulse and electron interaction can be described classically by the relativistic Newton-Lorentz equations

$$\frac{d\mathbf{P}}{dt} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{F}_r, \tag{25}$$

where $\mathbf{P}$ is electron momentum, $\mathbf{v}$ its velocity, $\mathbf{F}_r$ is the force induced by the radiation of the accelerated electron (which has usually been neglected in previous theoretical considerations), and $\mathbf{E}$ and $\mathbf{B}$ are the electromagnetic field components of laser pulse. In principle, by solving Eq. (25), we can predict the energy and trajectory of the electron to a very high accuracy, provided that we have a precise enough electromagnetic description for the laser pulse near the focal point and find a way of dealing with the radiation effect of the accelerated electron. Although the system of a laser pulse and single electron appears to be quite simple, it contains plenty of physical processes which are the basis of various acceleration schemes. An electron that initially stays near the focal point of the transversely polarized laser pulse will undergo an oscillatory motion induced by the transverse electric field. If the electric field is strong enough, the oscillation amplitude of the electron exceeds the focal spot radius, the restoring force acting on the electron decays exponentially, and it is scattered away from the focus, which is called ponderomotive scattering [23]. If the electron was not scattered out of the focal spot, besides the oscillatory motion it would also feel the force imposed by longitudinal electric field. Because the phase velocity of the longitudinal electric field E_z is greater than c, a relativistic electron with $v_z \sim c$ will undergo a phase change with respect to E_z and be decelerated. However, when the interacting distance of the electron with a laser beam is limited to the distance $l = r_0^2/2\pi\lambda$, the electron can be accelerated and get net energy from the laser beam. For ultrashort laser pulse there is an additional longitudinal ponderomotive force that arises from gradients in the laser pulse envelope [24], which has the form

$$F_z = -\frac{mc^2}{2\gamma}\frac{\partial}{\partial z}A^2. \tag{26}$$

Here, γ and m are the Lorentz factor and mass of the electron, respectively. The point of Eq. (26) is that, for two circularly polarized laser pulses, the phase velocity of F_z can be less than c under special conditions because of its nonlinearity; this can greatly extend the acceleration distance of a relativistic electron, which is the mechanism of the vacuum beat wave acceleration scheme. An electron which is initially injected sideways into the focal point of a tightly focused laser pulse can be reflected, or captured by the pulse, or transmitted through it. Which of these possibilities actually occurs for a given pulse is determined by the initial energy, position and incident angle of the electron. Numerical calculations show that the initial constant phase of the laser pulse hardly impacts upon the energy gain of the electron over the phase range $(0, \pi)$ [9]. The typical energy gain of an electron accelerated by a laser pulse in vacuum is $\sim$ hundreds of mega-electron-volts for a laser intensity $\sim 10^{20}$ W/cm^2 [8].

All the processes mentioned above have been studied theoretically, but these studies have only been tentative because a loose description of the electromagnetic field of laser pulse has usually been used with the radiation effect of the accelerated electron neglected. Laser pulse can now be focused down to a few microns, such as 5λ (λ is the wavelength and a typical value is 1 μm), or even to λ. For good performance a laser-driven electron accelerator should effectively utilize the maximum intensity of the electromagnetic field of the laser pulse and allow the interaction of the electrons and laser pulse to take place in the vicinity of the focal point, which is quite a small region. Laser-driven acceleration in vacuum is superior to laser-plasma acceleration in that, by solving Eq. (25), we can precisely determine the trajectory to within 0.1 μm (the energy gain of a single electron can also be determined accurately but this is less important because the speed of the electron is very close to c).

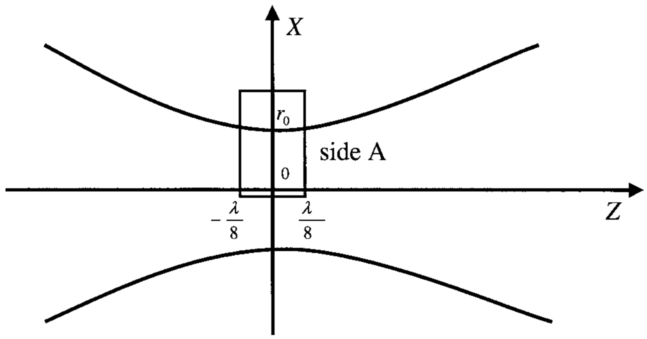

FIG. 1. Cross section of the box and laser pulse used in applying the integral form of Eq. (1).

Although the electric field can reach $\sim 10^{13}$ V/m, due to the restriction on accelerating distance, it is impossible to get very high energy from a single acceleration. To compete with and outperform the traditional accelerator, the repeated accelerations are essential. Laser-driven acceleration in vacuum allows repeated accelerations of charged particles and therefore has a promising application in the building of accelerators more powerful than present ones. This is because electron acceleration in vacuum by laser pulses allows for precise determination of the trajectory of the electrons. Consequently, the precise description of the electromagnetic field components of a tightly focused laser beam is an important contribution to the design of a powerful laser-driven electron accelerator. From Table I, when $s=0.02$ corresponding to the radius of a focal spot $r_0=3$ μm, for a laser pulse wavelength of $\lambda=1$ μm, the maximum ratio for the nine-order description of the electromagnetic components of the laser pulse is about 10^{-13}, which fulfills the requirements for the electromagnetic components of the laser pulse in controlling the motion of electron. So, our calculations of seventh- and ninth-order corrections of the components are significant. Of course, the radiation effect of the accelerated electron should also be considered.

V. LONGITUDINAL POLARIZATION

The existence of longitudinal electric field E_z for focused laser beam can be understood by applying the integral form of Eq. (1) to a half-round flat box with radius greater than the waist of laser pulse r_0, and the focal point of the pulse just inside (see Fig. 1). The integral is

$$\int E_z \mathrm{d}\sigma_z = -\int E_T \,\mathrm{d}\sigma_T. \tag{27}$$

By choosing a suitable time, the longitudinal electric field E_z reaches a maximum at one side of the box (side A in Fig. 1) and zero at the opposite side. Because the radius of the box is greater than r_0, E_y will not contribute to the integral on the right side of Eq. (27), and the only contribution is due to E_x at the side of the box in the plane $x=0$, so Eq. (27) can be written as

$$E_z \pi r_0^2/2 = E_x 2 r_0 \lambda/4.$$

We can then find the relationship $E_z \sim sE_x$, which is qualitatively right compared to the precise results Eq. (19) and Eq. (21). The appearance of E_z can be ascribed to the fact that the focused laser beam has a finite diameter, which invalidates the purely transverse picture of an electromagnetic plane wave in a vacuum.

What will happen when the vector potential is polarized longitudinally for a focused laser beam? In electromagnetic wave theory, the Lorentz invariance together with gauge invariance reduces the degree of freedom of the vector potential from 4 to 2. The fact that gauge transformation can eliminate the longitudinally polarized photon indicates that the longitudinal photon is unphysical and the real photon must be transversely polarized, which corresponds to the existence of only two kinds of transversely polarized photons in nature; in other words, the helicity of a real photon is just ±1. In contrast, a massive spin-1 particle has three polarization states. Therefore, to only choose the longitudinally polarized component of vector potential is forbidden. However, apart from the usual choice of a transversely polarized vector potential, there is another choice. For preserving the obvious covariance of theory, we consider four components of the vector potential first, and subsequently impose a condition that the contributions of scalar and longitudinal components of the vector potential cancel. For a plane-wave situation, this cancellation is exact, which is the main idea of the Gupta-Bleuler's indefinite metric quantization method in quantum field theory. For a focused laser, due to the fact that the scalar component $-(i/k)\boldsymbol{\nabla}\cdot\mathbf{A}\neq 0$, we can choose the vector part of a potential polarized along the propagation direction and expect that the cancellation of scalar and longitudinal photons will no longer exactly hold. Apart from the common part $E_0\psi_0 \exp(-i\zeta/s^2)$, by a rotation about the z axis the calculated electromagnetic field components of longitudinally polarized potential A are

$$\begin{aligned}E_x' = {} & (-2q\rho)s + q^2\rho(4i + 8q\rho^2 - 2iq^2\rho^4)s^3 + q^3\rho(-12 - 12iq\rho^2 - 34q^2\rho^4 + 12iq^3\rho^6 + q^4\rho^8)s^5 \\ & + q^4\rho\left(-48i + 48q\rho^2 + 24iq^2\rho^4 + 132q^3\rho^6 - 58iq^4\rho^8 - 8q^5\rho^{10} + \frac{i}{3}q^6\rho^{12}\right)s^7 \\ & + q^5\rho\left(240 + 4848iq\rho^2 + 3720q^2\rho^4 - 680iq^3\rho^6 - 526q^4\rho^8 + 254iq^5\rho^{10} + \frac{137}{3}q^6\rho^{12} - \frac{10i}{3}q^7\rho^{14} - \frac{1}{12}q^8\rho^{16}\right)s^9,\end{aligned}$$

$$\frac{H_y'}{\sqrt{\varepsilon}} = (-2q\rho)s + q^2\rho(-4i + 4q\rho^2 - 2iq^2\rho^4)s^3 + q^3\rho(12 + 12iq\rho^2 - 6q^2\rho^4 + 8iq^3\rho^6 + q^4\rho^8)s^5 + q^4\rho\left(48i - 48q\rho^2 - 24iq^2\rho^4 + 8q^3\rho^6 - 26iq^4\rho^8 - 6q^5\rho^{10} + \frac{i}{3}q^6\rho^{12}\right)s^7 + q^5\rho\left(-240 + 4368iq\rho^2 + 3960q^2\rho^4 - 600iq^3\rho^6 - 42q^4\rho^8 + 82iq^5\rho^{10} + \frac{79}{3}q^6\rho^{12} - \frac{8i}{3}q^7\rho^{14} - \frac{1}{12}q^8\rho^{16}\right)s^9,$$

$$E_z' = q(4i + 4q\rho^2)s^2 + q^2(-8 - 8iq\rho^2 - 20q^2\rho^4 + 4iq^3\rho^6)s^4 + q^3(-24i + 24q\rho^2 + 12iq^2\rho^4 + 76q^3\rho^6 - 26iq^4\rho^8 - 2q^5\rho^{10})s^6 + q^4\left(96 + 96iq\rho^2 - 48q^2\rho^4 - 16iq^3\rho^6 - 276q^4\rho^8 + 124iq^5\rho^{10} + \frac{50}{3}q^6\rho^{12} - \frac{2i}{3}q^7\rho^{14}\right)s^8 + q^5\left(480i - 480q\rho^2 - 32\,496iq^2\rho^4 - 12720q^3\rho^6 + 1620iq^4\rho^8 + 1068q^5\rho^{10} - \frac{1598i}{3}q^6\rho^{12} - \frac{286}{3}q^7\rho^{14} + \frac{41i}{6}q^8\rho 16 + \frac{1}{6}q^9\rho^{18}\right)s^{10}.$$

These electromagnetic field components are similar to those in the transversely polarized case in that E_x' is much larger than E_z', indicating that they, themselves, exhibit transverse polarization, and should be thought of as a real electromagnetic wave which can exist when the space it resides in is bounded in the transverse directions. However, the difference of the electromagnetic field between the two situations is also obvious. First, the electromagnetic field for longitudinal polarization situation has three components, as a result of the rotational symmetry about the z axis. For transverse polarization, the number of electromagnetic field components is five or six, showing that the two sets of electromagnetic wave solutions are not equivalent. Second, the magnitude of electromagnetic field components for longitudinal polarization case is higher in the order of the perturbative parameter s than the counterparts of transverse polarization. When the electromagnetic plane-wave beam is focused in vacuum, not only can it develop longitudinal field components near the focal point, but the scalar component of the vector potential does not exactly cancel the effects of the longitudinal component, which is different from the plane-wave case. When $s \to 0$, all the electromagnetic field components of the longitudinally polarized vector potential vanish.

As a by-product of our calculation, we have found two rules obeyed by all orders of the vector potential. If we change the variables from q,ρ to $q,\chi = q\rho^2$, we can see that every order of the vector potential can be put into the form

$$\psi_{2n} = q^n\left(\sum_{k=0}^{2n} a_k\chi^k\right)\psi_0, \tag{28}$$

which has two properties: (a) $a_1=0$ and (b) $a_k=0$ as $k>2n$, which can be proved from Eq. (12) for (a) and Eq. (11) for (b) using inductive reasoning. We think that these two rules are crucial for investigating the behavior of high-order corrections to the vector potential.

VI. SUMMARY

We have carefully derived seventh- and ninth-order corrections of a fundamental Gaussian beam, giving a significant improvement in accuracy as shown in Table I. The ninth-order description can be used to accurately simulate the interaction of electrons with an ultraintense laser pulse in vacuum, which is crucial for designing an accelerator driven by a laser in a vacuum capable of repeated electron accelerations. The nature of the electromagnetic field for the vector potential polarized along the propagating direction has also been discussed. The two properties possessed by all orders of the vector potential provide a different way to obtain perturbative solutions and make it possible to investigate the behavior of the solutions, which is of value for physical theories based on perturbative formalism. The numbers from Table I might indicate that the expansion is asymptotic.

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